DUSO Mathematics League 2014 - 2015

CONTEST #4.

SOLUTIONS

4 - 1. $\boxed{\mathbf{-1}}$ Substituting yields $a^2+b=5$ and $a^5+b=239$. Subtracting, we have $a^5-a^2=234$, which factors as $a^2(a^3-1)=9\cdot 26$, so $a^2=9$ and $a^3-1=26$. The value of a is 3 and the value of b is -4. Thus, $a+b=3+(-4)=\mathbf{-1}$.

4 - 2. $\boxed{\mathbf{30}}$ Consider square A. There are 3 color choices for A, and therefore there are 2 color choices for E. As for B and D, they are either the same color or different colors. If B and D are different colors, then there are 2 choices for B and 1 for D, and all that must be done now is to ascertain the number of choices for C. If B, D, and E are all different colors, there is no possible choice for either A or C. If, however, two of the three are the same, then there is 1 choice for C. Thus, if B and D are different colors, there are $3\cdot 2\cdot 2\cdot 1\cdot 1=12$ colorings. Now, consider the possibility that B and D are the same color. In this case, there are 2 choices for B and 1 for D. If B, D, and E are the same, then there are 2 choices for C (the remaining color or the color used for A) and therefore $3\cdot 2\cdot 2\cdot 1\cdot 1=12$ colorings. If, on the other hand, B and E are different colors, then there is only 1 choice for C and therefore $3\cdot 2\cdot 1\cdot 1\cdot 1=6$ colorings. The number of distinct possible colorings is $12+12+6=\mathbf{30}$.

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4 - 3. $\boxed{\mathbf{14}}$ Use the Power of a Point Theorem to find $OC=r$. $AE\cdot AC=AD\cdot AB$, so $4\cdot(2r+4)=8\cdot 16$ implies $r=\dfrac{32-4}{2}=\mathbf{14}$.

4 - 4. $\boxed{\mathbf{9}}$ The longer side is $\overline{AD}$, since the shorter altitude is drawn to it; let $AD=x$ and $AB=15-x$. Then, finding the area of the parallelogram in two different ways, equate $4x=6(15-x)$. Solve to obtain $4x=90-6x\to \mathbf{x}=\mathbf{9}$.

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4 - 5. $\boxed{\mathbf{y}=\mathbf{3}\sin\dfrac{\pi}{\mathbf{12}}\mathbf{x}+\mathbf{5}}$ The value of D is halfway between 8 and 2, so $D=\dfrac{8+2}{2}=5$. The value of A is the vertical distance between the maximum and the midline, so $A=8-5=3$. The period of the graph is $2\cdot(18-6)=24$, so $B=\dfrac{2\pi}{24}=\dfrac{\pi}{12}$. The equation of the graph is $y=3\sin\dfrac{\pi}{12}x+5$.

4 - 6. $\boxed{(\mathbf{3},\mathbf{-4})}$ Substituting $x-$ and $y-$coordinates into the general equation gives us three equations: $a+b+c=4$, $4a+2b+c=-2$, and $16a+4b+c=-2$. Subtracting the first two equations yields $3a+b=-6$. Subtracting the second two equations yields $12a+2b=0\to 6a+b=0$. Subtracting these yields $3a=6\to a=2\to b=-12\to c=14$. The equation is $y=2x^2-12x+14$, whose vertex is at $(\mathbf{3},\mathbf{-4})$.

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T-1. Compute all two-digit numbers such that the number is equal to twice the sum of its digits.
T-1Sol. $\boxed{18}$ Let the number be AB. Then, $10A + B = 2(A + B) \rightarrow 8A = B$. The only number that satisfies this is **18**.

T-2. There are two base-10 numbers for which the base-9 representation is ABC and the base-11 representation is CBA. Compute the sum of these two base-10 numbers in base 10.
T-2Sol. $\boxed{\mathbf{735}}$ The first sentence of the problem implies $81A + 9B + C = 121C + 11B + A$, which in turn implies $80A - 120C = 2B \rightarrow 20(2A - 3C) = B$. Since B is a multiple of 20 but $B < 9$, $B = 0$. Therefore, $2A - 3C = 0$. Since both A and C are less than 9, the only solutions are $A = 3$ and $C = 2$ or $A = 6$ and $C = 4$. The two base-9 numbers are 302 and 604, which convert to $3 \cdot 81 + 2 = 245$ and $6 \cdot 81 + 4 = 490$. Their sum is **735**.

T-3. Compute the values of x that solve the following equation: $(x + 5)^3 + (2x + 4)^3 = (3x + 9)^3$
T-3Sol. $\boxed{\{-\mathbf{5}, -\mathbf{3}, -\mathbf{2}\}}$ This equation is of the form $A^3 + B^3 = (A + B)^3$, which has solutions only if $A = 0$ or $B = 0$ or $A + B = 0$. Therefore, instead of expanding the brackets and proceeding to solve a difficult cubic, instead solve three linear equations to find $x + 5 = 0 \rightarrow x = -5$, $2x + 4 = 0 \rightarrow x = -2$, and $3x + 9 = 0 \rightarrow x = -3$. The solutions are $\{-\mathbf{5}, -\mathbf{3}, -\mathbf{2}\}$.

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